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Letter to the editor

Note on the cubic decreasing region of the Chebyshev method

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ABSTRACT

In this paper we reduce the two-dimensional cubic decreasing region considered in Hernandez and Salanova (2000) [1], [2] into one-dimensional region or interval for the Chebyshev method. It means that we find a simple sufficient condition for the semilocal convergence of the method.

1. Introduction

We consider a nonlinear equation

$$F(x) = 0. \tag{1}$$

Here $F : \Omega \subseteq X \to Y$ is a nonlinear, twice Frechet differentiable operator defined on a convex nonempty domain Ω and X and Y are Banach spaces. Chebyshev method is one of the best known third-order iterative process for solving the nonlinear equation (1) and it is given by

$$y_n = x_n - \Gamma_n F(x_n), \qquad \Gamma_n = F'(x_n)^{-1},$$
$$x_{n+1} = y_n - \frac{1}{2}\Gamma_n F''(x_n)(y_n - x_n)^2, \quad n = 0, 1, \ldots. \tag{2}$$

Let us assume that $F'(x_0)^{-1} \in \mathscr{L}(Y, X)$ exists for some $x_0 \in \Omega$, where $\mathscr{L}(Y, X)$ is the set of bounded linear operators from Y into X.
Moreover, we suppose that [1,2]

(c_1) $\|\Gamma_0\| \le \beta$,
(c_2) $\|y_0 - x_0\| = \|\Gamma_0 F(x_0)\| \le \eta$,
(c_3) $\|F''(x)\| \le M,\ x \in \Omega$,
(c_4) $\|F''(x) - F''(y)\| \le K\|x - y\|,\ x, y \in \Omega,\ K > 0$

and

$$a_0 = M\beta\eta > 0, \qquad b_0 = K\beta\eta^2 > 0. \tag{3}$$

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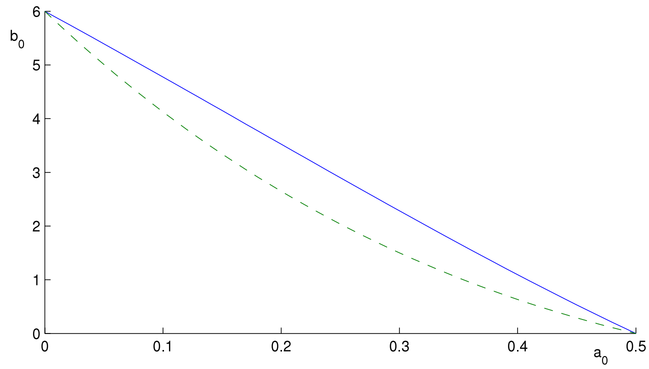

Fig. 1. Cubic decreasing regions.

In [1,2] the semilocal convergence theorem for the Chebyshev method was proven under the conditions (c_1)–(c_4) and

$$0 < a_0 < 1/2, \quad b_0 < \frac{3}{4}(1-2a_0)(a_0+2)(4-2a_0-a_0^2). \tag{4}$$

The cubic decreasing regions [1] of the Chebyshev method are presented in Fig. 1, where a_0 and b_0 are taken as coordinates.

The dotted line represents the curve

$$b_0 = \frac{6(1-2a_0)(4-2a_0-a_0^2)}{(2+a_0)^2}$$

obtained in [3] and the continuous line represents the curve

$$b_0 < \frac{3}{4}(1-2a_0)(a_0+2)(4-2a_0-a_0^2)$$

obtained in [1].

2. The cubic decreasing region

It is evident that the cubic decreasing region given in [1] is bigger, and therefore, the region of accessibility of the Chebyshev method has been increased. However the second inequality in (4) is not satisfied for $0 < a_0 < 1/2$ and it may hold only for the region $0 < a_0 < a_0^*$, where $a_0^* < 1/2$. Therefore, it is desirable to obtain a one-dimensional cubic decreasing region $0 < a_0 < a_0^*$ instead of the two-dimensional one given in (4) for the Chebyshev method. To this end, we proceed as follows: From (3) it is clear that there exists a closed relationship between b_0 and a_0. More precisely, we have

$$b_0 = c_n a_0^n, \quad \text{with } c_n = \frac{K}{M^n \beta^{n-1} \eta^{n-2}}, \quad n = 1, 2, \ldots.$$

When $n > 2$, the coefficient c_n will increase due to the small η and as a consequence, the cubic decreasing region will be reduced. From this, it is clear that it suffices to consider only the cases $n = 1$ and $n = 2$.

We first consider the case $n = 1$, i.e.

$$b_0 = c_1 a_0, \tag{5}$$

where

$$c_1 = \frac{K\eta}{M} > 0. \tag{6}$$

The second inequality in (4) leads as

$$\psi_1(a_0) = 2a_0^4 + 7a_0^3 - 4a_0^2 - \left(16 + \frac{4}{3}c_1\right) a_0 + 8 > 0. \tag{7}$$

Since $\psi_1(0) = 8 > 0$ and $\psi_1(1/2) = -\frac{2}{3}c_1 < 0$, there exists one value a_0^*, such that $\psi_1(a_0^*) = 0$ ($a_0^* < 1/2$). Differentiating (7) with respect to a_0 we obtain

$$\psi_1'(a_0) = 8a_0^3 + 21a_0^2 - 8a_0 - \left(16 + \frac{4}{3}c_1\right).$$

From this, it follows that $\psi_1'(a_0) < 0$ under $0 < a_0 < 1/2$, i.e. $\psi_1(a_0)$ is a decreasing function on the interval $0 < a_0 < 1/2$ and therefore, the inequality (7) holds for $0 < a_0 < a_0^*$.

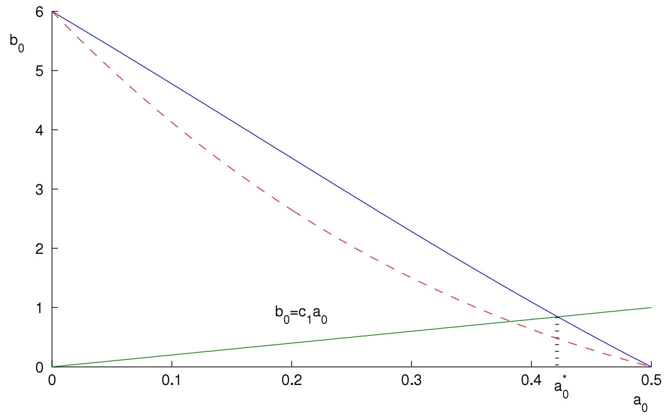

Fig. 2. The convergence region, $n = 1$ case.

We cannot find the point a_0^*, but we find the upper sharp bound for it:

$$a_0^* \le \hat{a}_0 = \frac{1}{2+c_1}.$$

Indeed, we have

$$\psi_1(\hat{a}_0) = \frac{c_1}{3z^4}(20z^3 - 12z - 3),$$

where $z = c_1 + 2$ and $20z^3 - 12z - 3 > 0$ for $2 \le z < \infty$. This means that

$$\psi_1(\hat{a}_0) > 0, \quad \text{when } c_1 > 0$$
$$\psi_1(\hat{a}_0) = 0, \quad \text{when } c_1 = 0.$$

Thus the one-dimensional (cubic decreasing) region for the Chebyshev method is

$$0 < a_0 < a_0^*, \tag{8}$$

where a_0^* is a root of the equation $\psi_1(a_0) = 0$, lying in $(0, 1/2)$, as shown in Fig. 2.

Now we consider the case of $n = 2$, i.e.

$$b_0 = c_2 a_0^2, \qquad c_2 = \frac{K}{M^2\beta} > 0.$$

In this case the second inequality in (4) leads to

$$\psi_2(a_0) = 2a_0^4 + 7a_0^3 - 4\left(1 + \frac{c_2}{3}\right)a_0^2 - 16a_0 + 8 > 0. \tag{9}$$

Since $\psi_2(0) = 8 > 0$, $\psi_2(1/2) = -\frac{1}{3}c_2 < 0$ and

$$\psi_2'(a_0) = 8a_0^3 + 21a_0^2 - 8\left(1 + \frac{c_2}{3}\right)a_0 - 16 < 0 \quad \text{for } 0 < a_0 < 1/2$$

then $\psi_2(a_0)$ is a decreasing function with respect to a_0 and therefore, there exists $\bar{a}_0 < 1/2$ such that $\psi_2(\bar{a}_0) = 0$. It means that $\psi_2(a_0) > 0$ when $0 < a_0 < \bar{a}_0$. As before, we can only find the upper bound for $\bar{a}_0$:

$$\bar{a}_0 \le \tilde{a}_0 = \frac{1}{2+c_2}.$$

Simple calculation gives

$$\psi_2(\tilde{a}_0) = \frac{1}{z^4}\left\{8z^4 - 16z^3 - 4\left(1 + \frac{c_2}{3}\right)z^2 + 7z + 2\right\} = \frac{z-2}{3z^4}(24z^3 - 4z^2 - 12z - 3),$$

where $z = 2 + c_2 \ge 2$.

The function $24z^3 - 4z^2 - 12z - 3$ is an increasing function of $z \in [2; +\infty)$. Therefore, we have

$$\psi_2(\tilde{a}_0) > 0, \quad \text{when } c_2 > 0$$
$$\psi_2(\tilde{a}_0) = 0, \quad \text{when } c_2 = 0.$$

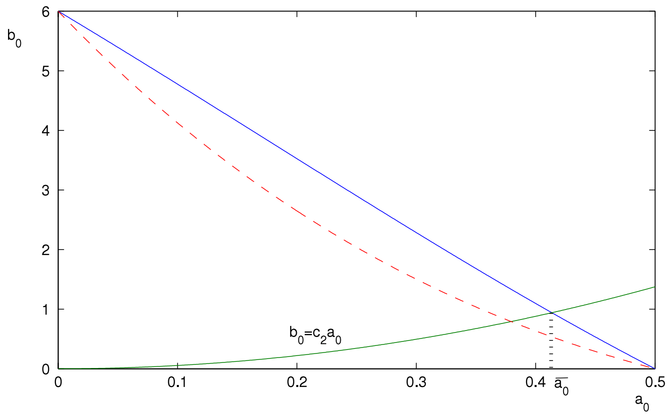

Fig. 3. The convergence region, $n = 2$ case.

Thus the inequality (4) holds under condition

$$0 < a_0 < \tilde{a_0}, \tag{10}$$

where $\tilde{a_0}$ is root of equation $\psi_2(a_0) = 0$, lying in $(0; 1/2)$, as shown in Fig. 3.

From (4), (9) and (10) we can conclude that the semilocal cubic convergence of the Chebyshev method holds under the conditions (c_1)–(c_4) and

$$0 < a_0 < \max\{\hat{a_0}, \tilde{a_0}\}.$$

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